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LETTER TO THE EDITOR

Dynamic equivalence of a two-dimensional quantum electron gas and a classical harmonic oscillator chain with an impurity mass

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Abstract. There is an exact equivalence in the time-dependent behaviour of a zero-temperature two-dimensional interacting electron gas at long wavelengths and a classical harmonic oscillator chain with one impurity mass. The mass difference $m-m_0$, where m_0 is the impurity mass, acts as the electron-electron interaction. Time evolution is asymmetric in $m-m_0$ about $m=m_0$.

Equivalence between two seemingly unrelated physical problems can often provide useful insight as shown by, for example, Lieb *et al* for an antiferromagnetic spin-$\frac{1}{2}$ XY chain and a free fermion model [1]. In a similar vein we show that there is an exact equivalence in the time-dependent behaviour of a quantum electron gas and a classical harmonic oscillator (HO) chain. According to the method of recurrence relations [2, 3], the time evolution of a dynamical variable, say A, depends on two dynamical parameters only: dimensionality $d=(f_0 f_1 \dots f_{d-1})$ and hypersurface $\sigma=(\Delta_1\Delta_2\dots\Delta_{d-1})$, $\Delta_\nu=\|f_\nu\|/\|f_{\nu-1}\|$, where the f_ν are the basis vectors which span S, the realised d-dimensional Hilbert space of $A(t)$, and $\|f_\nu\|$ is the norm of f_ν. Different systems thus can be dynamically equivalent (i.e. they can have the same autocorrelation function) if they have the same d and σ. We show here that a $T=0$ 2D electron gas at long wavelengths and a classical 1D NN coupled HO chain with one impurity mass both belong to the same dynamical class.

For the 2D electron gas there now exists a complete solution for the time evolution of density fluctuations as long wavelengths [4, 5]. The time evolution in a HO chain has been studied by several people, almost always via normal coordinates [6]. The recurrence relations analysis, however, is accomplished in the original lattice§. As a result, one can, for example, follow the delocalisation of an initial excitation from site to site. An impurity mass can greatly complicate the standard analysis, but in the recurrence relations analysis it causes only a minor modification of the realised Hilbert space.

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§ In the standard analysis one solves canonical equations in one form or another. In the recurrence relations analysis one obtains admissible solutions for a realised recurrence relation (RRII), which are the solutions of the original canonical equations.

For the 2D electron gas, we consider the density fluctuation operator ρ_k as a dynamical variable A, where k is the wavevector measured in units of the Fermi wave vector k_F. To order k, the realised Hilbert space of $\rho_k(t)$ is given by $d=\infty$ and $\sigma=(2s^{-1}\mu^2/4, \mu^2/4, \mu^2/4, \ldots)$, where $\mu=2k\varepsilon_F$, where ε_F is the Fermi energy, $s^{-1}=1+2\Gamma/\mu^2$, where $\Gamma=2\pi\rho e^2/m$, which is essentially the electron–electron interaction in 2D. Other symbols have their usual meaning, e.g. ρ is the electron number density. Since the interaction is repulsive, $1\leqslant s^{-1}\leqslant\infty$, where $s^{-1}=1$ and ∞ represent the ideal and non-ideal limits of the electron gas, respectively.

We now consider a chain of N HO with periodic boundary conditions, where N is an even number. Each spring has the same force constant κ. Let one oscillator have mass m_0 (designated as a tagged mass) and all others an identical mass m each. We introduce the parameter $\lambda=m/m_0$, where $\lambda=0$ and ∞ represent the heavy and light impurity mass limits, respectively. We choose $A=P_0$ the momentum of the tagged mass. The realised space of $P_0(t)$ is found to have the following properties:

$$d=N+1 \qquad \sigma=(2\lambda\kappa/m, \kappa/m, \kappa/m, \ldots, \kappa/m, 2\kappa/m).$$

For $\lambda=1$ (equal-mass limit) there is a front-end symmetry in σ. This symmetry is broken when $N\to\infty$; and this symmetry breaking gives rise to irreversibility.

When $N\to\infty$, the Hilbert space of $P_0(t)$ becomes exactly the same as the space of $\rho_k(t)$ up to some scale factors, which we fix by taking $\mu^2/4=\kappa/m$ and $s^{-1}=\lambda$. Hence the time evolution of P_0 in the equal-mass chain ($\lambda=1$) exactly corresponds to the time evolution of ρ_k in the ideal 2D electron gas ($s^{-1}=1$). Similarly the light impurity regime of the HO chain ($\lambda>1$) corresponds to the non-ideal or Coulomb electron gas (normal fermions). The heavy impurity regime ($\lambda<1$) would correspond to an attractively interacting electron gas ('abnormal fermions'). If such an 'abnormal fermion' system existed, it would just become bound at $\lambda=0$. There are no time evolutions known for this novel system, but one can obtain them from the heavy impurity regime of the HO chain. Thus we seek a general solution.

Now $\rho_k(t)$ and $P_0(t)$ satisfy, respectively, the Heisenberg and canonical equations. Equivalently, both dynamical variables, say A, satisfy the generalised Langevin equation (GLE):

$$\frac{dA(t)}{dt}+\int_0^t dt' M(t-t')A(t')=F(t) \tag{1}$$

where $M(t)$ is the memory function and $F(t)$ is the random force [2]. The method of recurrence relations formally solves the GLE by giving $A(t)$, and $F(t)$, an orthogonal expansion in the f_ν, i.e.

$$A(t)=\sum_{\nu=0}^{d-1} a_\nu(t) f_\nu \tag{2}$$

$$F(t)=\sum_{\nu=1}^{d-1} b_\nu(t) f_\nu \tag{3}$$

where the a_ν and the b_ν are certain correlation functions of time. For the electron gas, for example, $a_0(t)=(\rho_k(t), \rho_k)/(\rho_k, \rho_k)$, where the inner product means the Kubo scalar product [2]. For the HO chain, $a_0(t)=\langle P_0(t)P_0\rangle/\langle P_0^2\rangle$, where $\langle\ldots\rangle$ means a classical ensemble average. The space of the random force $F(t)$, say S_1, is a subspace of S, the space of $A(t)$, and its hypersurface is denoted by $\sigma_1=(\Delta_2\Delta_3\ldots\Delta_{d-1})$. Also, $M(t)=\Delta_1 b_1(t)$.

In the ideal or equal-mass limit ($s^{-1} = \lambda = 1$), one has $\sigma = (2111\ldots)$ and $\sigma_1 = (111\ldots)$ up to some common scale factors, set to unity here. They imply that

$$a_\nu(t) = 2^\nu \mu^{-\nu} J_\nu(\mu t) \qquad \nu \geqslant 0 \tag{4}$$

$$b_\nu(t) = 2^\nu \mu^{-\nu} \nu J_\nu(\mu t)/t \qquad \nu \geqslant 1 \tag{5}$$

where J_ν is the Bessel function of order ν†. The interaction or impurity mass changes σ but not σ_1, hence $a_\nu(t)$ but not $b_\nu(t)$. We take advantage of a constant σ_1 to obtain a general solution for $a_\nu(t)$ via a connecting relation [2]

$$a_0(z) = (z + \Delta_1 b_1(z))^{-1} \tag{6}$$

where $a_0(z)$ and $b_1(z)$ are, respectively, the Laplace transforms of $a_0(t)$ and $b_1(t)$. From (5), $b_1(z) = 2\mu^{-2}(\sqrt{z^2+\mu^2} - z)$. Hence

$$a_0(t) = \frac{1}{2\pi \mathrm{i}} \int_c \mathrm{d}z \exp(zt) a_0(z) = \frac{1}{2\pi \mathrm{i}\lambda} \int_c \mathrm{d}z \frac{\exp(zt)}{pz + \sqrt{z^2+\mu^2}} \tag{7}$$

where $p = \lambda^{-1} - 1$. Given $a_0(t)$ from (7), we can obtain all other $a_\nu(t)$ by a recurrence relation (RRII). Hence, the time evolution of P_0 can be completely characterised by this analysis‡.

We see in (7) that there are, in addition to a branch cut, singularities in each of two branches, which are determined by the sign of p§. To obtain these singularities explicitly, we write the relevant part of (7) as

$$\frac{\sqrt{z^2+\mu^2} - pz}{\alpha(z^2 + \alpha^{-1}\mu^2)} \tag{7a}$$

where $\alpha = 1 - p^2 = (2\lambda - 1)/\lambda^2$. Clearly the locations of isolated poles in a given branch (i.e. a given sign of p) depend on the sign and size of α^{-1}. In figure 1(a), α^{-1} is plotted against λ, with p superimposed therein.

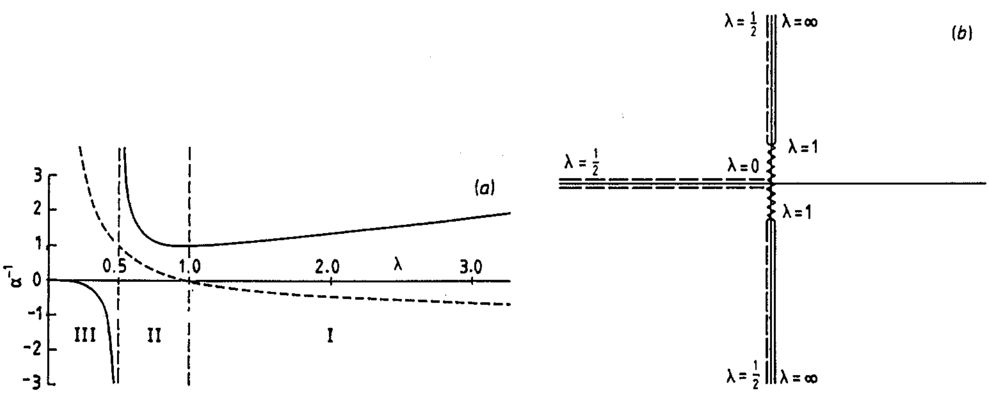

Figure 1. (a) The solid curve is a plot of α^{-1} against λ. The function $p = \lambda^{-1} - 1$ is plotted as a broken curve. (b) Poles in the first branch (full lines), poles in the second branch (broken lines) and the branch cut (zigzag).

† The solution $a_0(t) = J_0(\mu t)$ is well known. See [4, 6].

‡ The basis vectors f_ν can be obtained by RRI, given that $f_0 = P_0$ or ρ_k. See [2].

§ The denominator $pz + \sqrt{z^2+\mu^2}$ is closely related to a function which appears in the Joukowski transformation in the theory of aerofoils. See, e.g., [7].

There are three distinct regions. In region I we have $1<\lambda<\infty$, where $\alpha^{-1}>1$ and $p<0$. There are a pair of poles on the imaginary axis beyond the branch points on the first or physical sheet†. In region II, $\frac{1}{2}<\lambda<1$, where $\alpha^{-1}>1$ and $p>0$. There are also the same poles of region I but on the second or non-physical sheet. In region III, $0<\lambda<\frac{1}{2}$, where $\alpha^{-1}<0$ and $p>0$. There is one pole on the negative real axis, also on the non-physical sheet. In figure 1(*b*), these poles are illustrated as a function of λ for a given branch.

Contributions to (7) from isolated poles are limited to those on the physical sheet, i.e. the poles of region I‡. Contributions from the branch cut are, however, similar in each of the three regions depending only on α^{-1}. They are identical in regions I and II since both regions have the same α^{-1}. Region III can be subdivided into IIIA ($\alpha^{-1}>-1$) and IIIB ($\alpha^{-1}<-1$). In subregion IIIA the solution of regions I and II applies with $-\alpha$. In subregion IIIB there is a different solution.

In region I, the complete solution is

$$a_0(t)=(2|p|/(1+|p|)\cos\Omega t+\sum_{n=0}^{\infty}(-\alpha)^n(\partial/\partial\mu t)^{2n}J_1(\mu t)/\mu t \tag{8}$$

where $\Omega=(\alpha^{-1}\mu^2)^{1/2}$. The above solution without the cosine term is also the solution in regions II and IIIA as previously noted.

In subregion IIIB, the solution is

$$a_0(t)=(\pi\lambda|\alpha|)^{-1}\sum_{n=1}^{\infty}\beta^n\Gamma(n+\tfrac{1}{2})J_n(\mu t)/(\mu t/2)^n \tag{9}$$

where $\beta=|\alpha|/(1+|\alpha|)=(1-2\lambda)/(1-\lambda)^2$. In region III, this new parameter β ranges from 0 to 1. Thus, the above solution (9) also applies to subregion IIIA. When $\beta=1$, (9) is divergent except when $t=0$, wherein $a_0(t=0)=1$. Thus, no expansion about $\lambda=0$ is possible for an arbitrary time§.

We shall briefly discuss the equivalence aspect of our solutions as λ varies from ∞ to 0. When $\lambda=\infty$ ($m=\infty$, $m_0<\infty$), one gets $a_0(t)=\cos\Omega_\infty t$, $\Omega_\infty=(2\kappa/m_0)^{1/2}$, i.e. the tagged mass bound to two stationary walls‖. In the electron gas it is exactly the condition in which single-particle motions are either frozen or completely overwhelmed by the plasma oscillation (strong-coupling limit). As $\lambda\to 1$, the initial simple oscillatory motion becomes perturbed by the 'moving walls'. In the equal-mass limit, it disappears entirely as when the electrons no longer interact. But in this limit the motion of the tagged mass is indistinguishable from the motions of other masses in the chain. Hence, it is more nearly in phase and its autocorrelation function decays more slowly, i.e. $a_0(t\to\infty)\sim t^{-1/2}\cos(\mu t-\pi/4)$. It is just the behaviour of long-lived electron–hole pair excitations, existing very near the Fermi surface in the weak-coupling limit [9].

As $\lambda\to 0$, the motion of the tagged mass begins to go out of phase. For the electrons (now abnormal fermions), the excitations tend to be localised owing to an attractive

† The physical sheet is one on which the solutions obtained for $a_\nu(t)$ are admissible, i.e. they satisfy RRII. Necessary conditions are $a_0(t=0)=1$ and $\mathrm{d}a_0(t=0)/\mathrm{d}t=0$. This is the only branch which is physically relevant. See [8].

‡ For the electron gas these are just the plasma poles, with the branch cut representing single-particle excitations.

§ An expansion solution is possible only under some special conditions, e.g. $\lambda t<1$. In contrast, (8) is a general expansion about $\lambda^{-1}=0$ and converges even at $\alpha=1$.

‖ One can also obtain $\lambda=\infty$ by taking $m<\infty$ and $m_0=0$. For this case the isolated poles diappear, i.e. no cosine term in (8), and $a_0(t)=2J_1/t$, the same as that for $\lambda=\frac{1}{2}$.

interaction, thus precluding any formation of plasma-like collective modes. For $0 < \lambda < 1$, the motions of the tagged mass represent the scattering states of the abnormal fermions, which therefore cannot be continuously changed into the state of $\lambda = 0$, a bound state. Time evolutions evidently are asymmetric in the mass difference $m - m_0$ about $m = m_0$, which is equivalent to the electron-electron interaction†.

Finally, a one-impurity HO chain has a subspace S_1 independent of λ. This is just the condition that the generalised RPA theory of an electron gas [9] is exactly valid. Hence, selectively adding more impurity masses to the chain is similar to systematically correcting the RPA. One particular limit of a homogeneous multi-impurity chain is a diatomic chain. This limiting process, in effect, forms new branch cuts by extending the isolated poles of a one-impurity chain. The two chains differ in time evolution to the extent of this difference in the analytic structure [11].

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† The HO chain is dynamically not equivalent to the electron gas in $D = 1$ and 3. If $D = 1$, $d = 2$. If $D = 3$, $d = \infty$, but $\sigma = (4s^{-1}/3, \frac{16}{15}, \frac{36}{35}, \ldots 4\nu^2/(4\nu^2 - 1), \ldots)$ in units of $\mu^2/4 = 1$. See [10].